FP3 Specimen Paper Prime 2

1. $A = \begin{pmatrix} 7 & 6 \\ 6 & 2 \end{pmatrix}$, $A - \lambda I = \begin{pmatrix} 7-\lambda & 6 \\ 6 & 2-\lambda \end{pmatrix}$

$$\det(A - \lambda I) = (7-\lambda)(2-\lambda) - 36$$
$$= -22 - 9\lambda + \lambda^2 = 0$$

$\det(A - \lambda I) = 0 \Rightarrow \lambda^2 - 9\lambda - 22 = 0$

$\therefore (\lambda - 11)(\lambda + 2) = 0$

E. val :

$\Rightarrow$ $\lambda = -2$

$\lambda = 11$ //

2. $\frac{9}{2}(e^x + e^{-x}) - \frac{6}{2}(e^x - e^{-x}) = 7$

$\therefore \frac{9}{2}e^x + \frac{9}{2}e^{-x} - 3e^x + 3e^{-x} = 7$

$\therefore \frac{3}{2}e^x + \frac{15}{2}e^{-x} = 7$

$\Rightarrow 3e^x + 15e^{-x} = 14$

$\Rightarrow 3e^{2x} + 15 = 14e^x$

$\Rightarrow 3e^{2x} - 14e^x + 15 = 0$

$(e^x - 3)(3e^x - 5) = 0$ $\qquad e^x = \frac{5}{3}$

$\Downarrow$

$\therefore e^x = 3 \Rightarrow x = \ln 3$ // $\qquad x = \ln\frac{5}{3}$ //

3. Let length $= s$

$$S = \int_0^{2\pi} \sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2}\, \partial t$$

$$x = a(t - \sin t) \Rightarrow \frac{\partial x}{\partial t} = a(1 - \cos t)$$

$$y = a(1 - \cos t) \Rightarrow \frac{\partial y}{\partial t} = a(\sin t)$$

$$\therefore \quad S = \int_0^{2\pi} \sqrt{a^2(1-\cos t)^2 + a^2 \sin^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{a^2\left(1 - 2\cos t + \cos^2 t + \sin^2 t\right)}\, dt$$

$$= \int_0^{2\pi} \sqrt{a^2(2 - 2\cos t)}\, \partial t$$

$$= \int_0^{2\pi} \sqrt{2a^2(1 - \cos t)}\, dt$$

$\cos 2t = \cos^2 t - \sin^2 t$

$\cos t = \cos^2 \frac{t}{2} - \sin^2 \frac{t}{2}$

$\cos^2 \frac{t}{2} - (1 - \cos^2 \frac{t}{2}$

$\cos t = 2\cos^2 \frac{t}{2} - 1$

$$= \sqrt{2}\, a \int_0^{2\pi} \sqrt{1 - \cos t}\, dt$$

$$= a\sqrt{2} \int_0^{2\pi} \sqrt{2\cos^2 \frac{t}{2}}\, \partial t = a\sqrt{2}\sqrt{2} \int \sin \frac{t}{2}\, dt$$

$$= 2a\left[-2\cos \frac{t}{2}\right]^{2\pi} = 8a$$

$$= 2a\,(2 - -2)$$
$$= 8a$$
$$s = 8a$$

4. $\cancel{\int (x^2+4)^{1/2}\, dx}$

$\cancel{u = (x^2+4)^{1/2} \quad u' = \frac{1}{2}(x^2+4)^{-1/2} \times 2x = x\sqrt{x^2+4}}$

$\cancel{v' = 1 \quad v = x}$

$\cancel{x\sqrt{(x^2+4)} - \int x^2\sqrt{x^2+4}}$

$\cancel{s^2 + c^2 =}$

$c^2 - s^2 = 1$

$c^2 = 1 + s^2 = s^2 + 1$

4. $\int (x^2+4)^{1/2}\, dx$

$x = 2\sinh\theta \qquad \frac{x}{2} = \sinh\theta$

$x^2 = 4\sinh^2\theta \qquad \theta = \operatorname{arsinh}\frac{x}{2}$

$= \ln\left(\frac{x}{2} + \sqrt{\frac{x^2}{4}+1}\right)$

$\frac{dx}{d\theta} = 2\cosh\theta$

$dx = 2\cosh\theta\, d\theta$

$$= \int (4\sinh^2\theta + 4)^{1/2} \cdot 2\cosh\theta\, d\theta$$

$$= 2\int \sqrt{4(\sinh^2\theta+1)} \cdot \cosh\theta\, d\theta$$

$\cos 2x = \cos^2 - \sin^2$

$$= 2\int 2\cosh^2\theta\, d\theta$$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

$\cosh 2x = 2\cosh^2 x - 1$

$$= 4\int \cosh^2\theta\, d\theta$$

$\frac{\cosh(2x)+1}{2} = \cosh^2 x$

$$= \frac{4}{2}\int \cosh(2\theta) + 1\, d\theta$$

$\sinh 2\theta = 2\sinh\theta\cosh\theta$

$= x\sqrt{1+\sinh^2\theta}$

$= x\sqrt{1+\frac{x^2}{4}}$

$$= 2\left[\frac{1}{2}\sinh(2\theta) + \theta\right] + C$$

$$= \sinh 2\theta + 2\theta + C = x\sqrt{1+\frac{x^2}{4}} + 2\ln\left(\frac{x}{2} + \sqrt{\frac{x^2}{4}+1}\right) + C$$

5 (a) $y = \arcsin x$

$\therefore \quad \sin y = x$ $\qquad\qquad \cos y \equiv \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$

$\therefore \quad \cos y \dfrac{\partial y}{\partial x} = 1$

$\therefore \quad \dfrac{\partial y}{\partial x} = \dfrac{1}{\cos(y)}$

$\therefore \quad \dfrac{\partial y}{\partial x} = \dfrac{1}{\sqrt{1-\sin^2 y}} = \dfrac{1}{\sqrt{1-x^2}}$ //

$(1-x^2)^{-1/2}$

(b) $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{1}{\sqrt{1-x^2}}\right) = -\dfrac{1}{2}(1-x^2)^{-\frac{3}{2}}(-2x)$

$= x(1-x^2)^{-3/2}$

$\dfrac{\partial^2 y}{\partial x^2} = \dfrac{x}{\left(\sqrt{1-x^2}\right)^3}$

$\therefore \quad (1-x^2)\dfrac{\partial^2 y}{\partial x^2} = x(1-x^2)^{-3/2}(1-x^2)$

$= x(1-x^2)^{-1/2}$

$= x \cdot \dfrac{1}{\sqrt{1-x^2}}$

$= x \cdot \dfrac{\partial y}{\partial x}$

$\Rightarrow \quad (1-x^2)\dfrac{\partial^2 y}{\partial x^2} - x\dfrac{\partial y}{\partial x} = 0$ //

6.(a).

$$I_n = \int_0^{\pi/2} x^n \sin x \, \partial x$$

Let $u = x^n \qquad u' = n x^{n-1}$

$v' = \sin x \qquad v = -\cos x$.

$$\therefore \quad I_n = \left[-x^n \cos x\right]_0^{\frac{\pi}{2}} + n \int_0^{\pi/2} x^{n-1} \cos x \, \partial x$$

$$= n \int_0^{\pi/2} x^{n-1} \cos x \, dx$$

$u = x^{n-1} \qquad u' = (n-1) x^{n-2}$

$v' = \cos x \qquad v = \sin x$

$$\therefore \quad I_n = n \left(\left[x^{n-1} \sin x\right]_0^{\pi/2} - (n-1) \int_0^{\pi/2} x^{n-2} \sin x \right)$$

$$= n \left(\left(\frac{\pi}{2}\right)^{n-1} - (n-1) I_{n-2} \right)$$

$$= n \left(\frac{\pi}{2}\right)^{n-1} - n(n-1) I_{n-2}$$

as required.

(b) $I_3 = 3\left(\frac{\pi}{2}\right)^2 - 6\, I_1$

$u = x \quad u' = 1$

$v' = \sin x \quad v = -\cos x$

$$= \frac{3}{4}\pi^2 - 6\int_0^{\pi/2} x \sin x \, dx$$

$$= \frac{3}{4}\pi^2 - 6\left[\left[-x\cos x\right]_0^{\pi/2} + \int_0^{\pi/2} \cos x \, dx\right)$$

$$= \frac{3\pi^2}{4} - 6\left(\left[\sin x\right]_0^{\pi/2}\right)$$

$$= \frac{3\pi^2}{4} - 6(1-0)$$

$$= \frac{3\pi^2}{4} - 6$$

7(a). $\begin{pmatrix} 1 & x & -1 \\ 3 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}$

$$\det[A(x)] = \cancel{-2(} -2 - x(-2) - 3 = \underline{\underline{2x-5}}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$M = \begin{pmatrix} -2 & -2 & 3 \\ 1 & 1 & 1-x \\ 2x & 5 & -3x \end{pmatrix}$$

$$C = \begin{pmatrix} -2 & 2 & 3 \\ -1 & 1 & x-1 \\ 2x & -5 & -3x \end{pmatrix}$$

$$C^{T} = \begin{pmatrix} -2 & -1 & 2x \\ 2 & 1 & -5 \\ 3 & x-1 & -3x \end{pmatrix}$$

$$\therefore \left[A(x)\right]^{-1} = \frac{1}{2x-5}\begin{pmatrix} -2 & -1 & 2x \\ 2 & 1 & -5 \\ 3 & x-1 & -3x \end{pmatrix}$$

(b) $\begin{pmatrix} 1 & 3 & -1 \\ 3 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

$\therefore \begin{pmatrix} p+3q-r \\ 3p+2r \\ p+q \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

$\therefore \quad p+3q-r = 2$

$\Rightarrow r = p+3q-2$

$\therefore \quad 3p+2r = 3p+2p+6q-4 = 5p+6q-4$

$5p+6q-4 = 3 \Rightarrow 5p+6q = 7$

$p+q = 4$

$\therefore \quad 5p+5q = 20$

$5p+6q = 7 \Rightarrow 5p+5q+q = 7$

$\Rightarrow 20+q = 7$

$\Rightarrow q = -13$

$\therefore 5p = 20-5(-13) = 85$

$\Rightarrow p = 17$

$\cancel{2r}$ $r = \dfrac{3-3p}{2} = -24$

$\therefore \quad p = 17$

$q = -13$

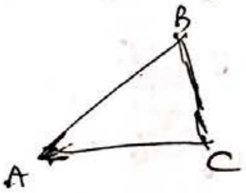
$r = -24$

8.
(a) $\vec{AB} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}$

$\vec{AC} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$

$\therefore \vec{AB} \times \vec{AC} = \times \quad \begin{matrix} -1 & & 3 & & -1 & & -1 & & 3 & & -1 \\ & & & \times & & \times & & \times & & & \\ -1 & & 3 & & 1 & & -1 & & 3 & & 1 \end{matrix}$

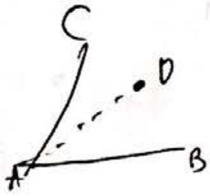
$= \begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix}$



$\therefore$ Area $= \left| \frac{1}{2} \begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix} \right| = \left\| \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \right\| = \sqrt{3^2 + 1}$

$= \sqrt{10}$

(b)



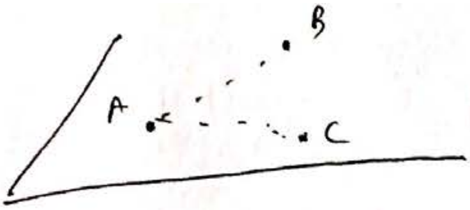
$AD = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix}$

Area $= \frac{1}{6} \left| \vec{AD} \cdot (\vec{AB} \times \vec{AC}) \right|$

$= \frac{1}{6} \left| \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix} \right|$

$= \frac{1}{6} \left| -12 + 8 \right| = \frac{2}{3}$

(c)



$\underset{\sim}{n} = AB \times AC = \begin{pmatrix}6\\2\\0\end{pmatrix}$

$= \begin{pmatrix}0\\4\\1\end{pmatrix} + \lambda($

$D\begin{pmatrix}0\\4\\1\end{pmatrix}$

$\Rightarrow \underset{\sim}{r}\cdot\begin{pmatrix}6\\2\\0\end{pmatrix} = \begin{pmatrix}2\\0\\1\end{pmatrix}\cdot\begin{pmatrix}6\\2\\0\end{pmatrix}$

$\underset{\sim}{r}\cdot\begin{pmatrix}6\\2\\0\end{pmatrix} = 12$

$\frac{1}{\sqrt{36+4}}$

$\hat{\underset{\sim}{n}} = \frac{1}{\sqrt{40}}\begin{pmatrix}6\\2\\0\end{pmatrix}$

$\underset{\sim}{r}\cdot\underset{\sim}{n} = 12$

$D = \begin{pmatrix}0\\4\\1\end{pmatrix}$

$\Rightarrow$ eqn of plane:

$6x+2y=12$

$3x+y=6$

$3x+y-6=0$

$\therefore$ distance $= \left|\frac{3(0)+4-6}{\sqrt{3^2+1^2}}\right|$

$= \left|\frac{-2}{\sqrt{10}}\right| = \frac{\sqrt{10}}{5}$

9(a). $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore \frac{2}{a^2}x - \frac{2}{b^2}y\frac{\partial y}{\partial x} = 0$

$\therefore \frac{2}{a^2}x = \frac{2}{b^2}y\frac{\partial y}{\partial x}$

$\frac{1}{a^2}x = \frac{1}{b^2}y\frac{\partial y}{\partial x}$

$\frac{b^2}{a^2}\frac{x}{y} = \frac{\partial y}{\partial x}$

$\frac{\frac{1}{\cos}}{\frac{\sin}{\cos}}$

$\left(\frac{\partial y}{\partial x}\right)_{\substack{x = a\sec\theta \\ y = b\tan\theta}} = \frac{b^2}{a^2} \cdot \frac{a\sec\theta}{b\tan\theta} = \frac{b^{\cancel{2}}\cancel{a}}{a^{\cancel{2}}\cancel{b}}\operatorname{cosec}\theta$

$= \frac{b}{a}\operatorname{cosec}\theta$

$\therefore m_N = \frac{-1}{\frac{b}{a}\operatorname{cosec}\theta} = -\frac{1}{\frac{b}{a}}\sin\theta = -\frac{a}{b}\sin\theta$

$\therefore y - b\tan\theta = -\frac{a}{b}\sin\theta(x - a\sec\theta)$

$\therefore by - b^2\tan\theta = -a\sin\theta(x - a\sec\theta)$

$\therefore by - b^2\tan\theta = -ax\sin\theta + a^2\sin\theta\sec\theta$

$\therefore by\tan x\sin\theta = a^2\frac{\sin\theta}{\cos\theta} + b^2\tan\theta$

$by\tan x\sin\theta = a^2\tan\theta + b^2\tan\theta$

as required

$\therefore by\tan x\sin\theta = (a^2+b^2)\tan\theta$

(b) when $x=0$

~~AT~~ $by = (a^2+b^2)\tan\theta$

$\Rightarrow y = \frac{a^2+b^2}{b}\tan\theta$

$\left(0, \frac{a^2+b^2}{b}\tan\theta\right)$

$\frac{\frac{1}{\cos}}{\frac{\sin}{\cos}}$

when $y=0$

$ax\sin\theta = (a^2+b^2)\tan\theta$

$\therefore x = \frac{(a^2+b^2)}{a}\sec\theta$

$\therefore \left(\frac{a^2+b^2}{a}\sec\theta, 0\right)$

$\therefore M = \left(\overset{x=}{\frac{a^2+b^2}{2a}\sec\theta}, \overset{y=}{\frac{a^2+b^2}{2b}\tan\theta}\right)$

$\frac{s^2}{c^2}+\frac{c^2}{c^2} = \frac{1}{c^2}$

$t^2+1 = \sec^2$

$X = \frac{a^2+b^2}{2a}\sec\theta \qquad Y = \frac{a^2+b^2}{2b}\tan\theta$

$\sec\theta = \sqrt{\tan^2\theta+1}$ & $\frac{2bY}{a^2+b^2} = \tan\theta$

$\therefore X = \frac{a^2+b^2}{2a}\sqrt{\tan^2\theta+1} = \frac{a^2+b^2}{2a}\sqrt{\frac{4b^2Y^2}{(a^2+b^2)^2}+1}$

$\therefore X^2 = \frac{(a^2+b^2)^2}{4a^2}\left(\frac{4b^2Y^2}{(a^2+b^2)^2}+1\right)$

$$x^2 = \frac{4b^2y^2}{4a^2} + \frac{(a^2+b^2)^2}{4a^2}$$

$$x = \frac{b^2}{a^2}y^2 + \frac{(a^2+b^2)^2}{4a^2} \qquad \times 4a^2$$

$$x = \frac{b^2}{a^2}y^2 + \frac{(a^2+b^2)^2}{4a^2}$$

$$4a^2x^2 = 4b^2y^2 + (a^2+b^2)^2$$

$$\therefore 4a^2x^2 - 4b^2y^2 = (a^2+b^2)^2$$